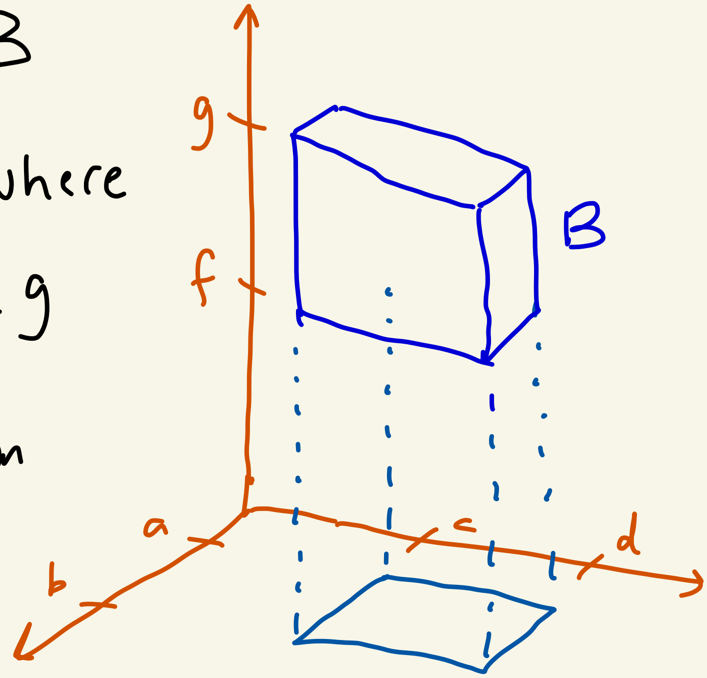
Topic 6 - Triple Integrals

Suppose that the box B is described by all (x,y,z) points in 3d where

$$a \le x \le b,\ c \le y \le d,\ f \le z \le g$$



Let $f(x,y,z)$ be a function defined on B.

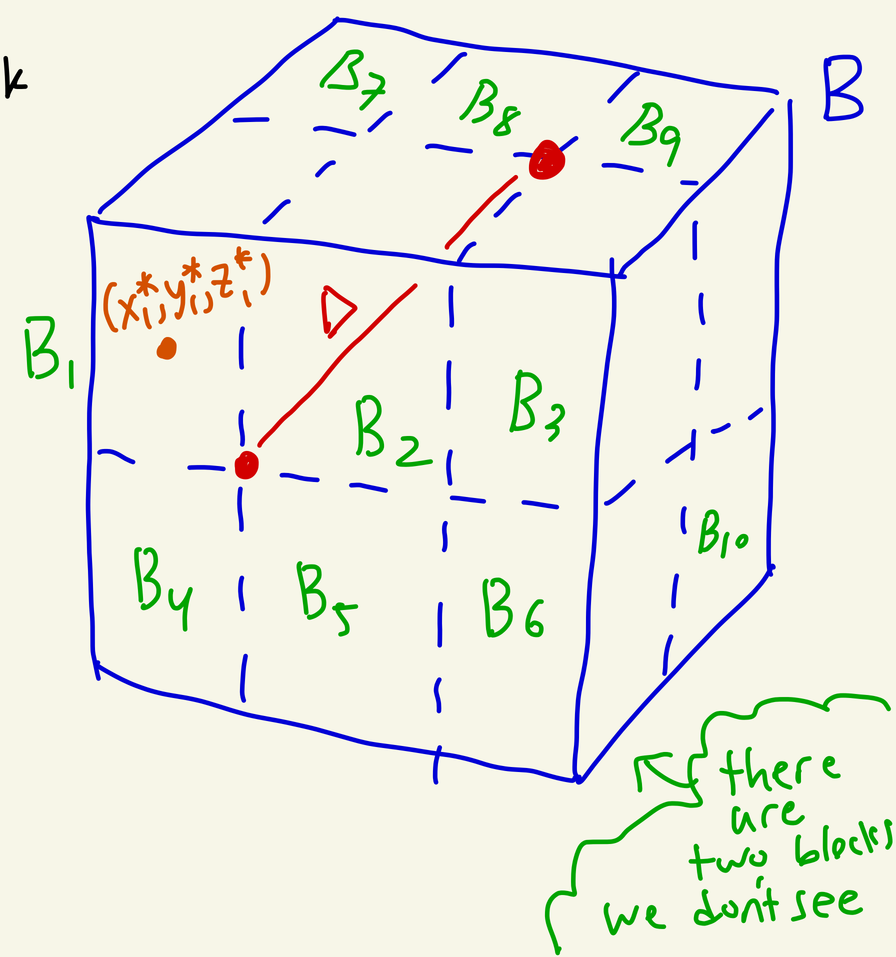
$\boxed{\underline{\text{Ex:}}\ f(x,y,z) = x^2 y + z^3}$

We want to define the integral of f over B.
We sub-divide B into n sub-boxes B_k.
First we sub-divide B using planes parallel to the xy-plane, yz-plane, and xz-plane.

Suppose each block B_k has volume ΔV_k.

For each k, let (x_k^*, y_k^*, z_k^*) be a point in B_k.

Let Δ be the length of the largest diagonal amongst the B_k.



Then we can create the following weighted sum

$$\underbrace{\sum_{k=1}^{n}}_{\text{add up all the weighted volumes on the right}} \underbrace{f(x_k^*, y_k^*, z_k^*)}\cdot \underbrace{\Delta V_k}_{\text{take the volume of each } B_k \text{ and weight it by this factor}}$$

Now we let $n \to \infty$ [# boxes $\to \infty$]
and $\Delta \to 0$ [size of the boxes $\to 0$]
to define the triple integral:

$$\boxed{\iiint_B f(x,y,z)\, dV = \lim_{\substack{n\to\infty \\ \Delta\to 0}} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*) \cdot \Delta V_k}$$

if the limit exists.

Note: You can think of the integral as "adding up" all the values of $f(x,y,z)$ over E in some sense.

Theorem: If $f(x,y,z)$ is continuous on the box B defined by all (x,y,z) with $a \le x \le b$, $c \le y \le d$, $f \le z \le g$ then

$$\iiint_B f(x,y,z)\,dV = \int_f^g \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Fact: You can re-order the dx, dy, dz in any order that you want. Just make sure to change the integral bounds also

Ex: Calculate $\iiint_B xyz^2\,dV$ where B is defined by $0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 1.$

$$\iiint_B xyz^2\,dV = \int_0^1 \int_0^2 \underbrace{\left(\int_0^1 xyz^2\,dx \right)}_{\substack{\text{integrate with respect to } x \\ \text{treat } y \text{ and } z \text{ as constants}}} dy\,dz$$

$$= \int_0^1 \int_0^2 \underbrace{\left(\frac{x^2}{2} yz^2 \Big|_{x=0}^{x=1} \right)}_{\text{plug in } x=1 \text{ and } x=0 \text{ and subtract}} dy\,dz$$

$$= \int_0^1 \int_0^2 \left[\frac{1}{2} yz^2 - 0 \right] dy\,dz$$

$$= \int_0^1 \underbrace{\left(\int_0^2 \frac{1}{2} yz^2\,dy \right)}_{\substack{\text{integrate with respect to } y \\ \text{treat } z \text{ as a constant}}} dz$$

$$= \int_0^1 \left(\frac{1}{4} y^2 z^2 \Big|_{y=0}^{y=2} \right) dz$$

$$= \int_0^1 \left(\frac{1}{4} \cdot 2^2 \cdot z^2 - \frac{1}{4} \cdot 0^2 \cdot z^2 \right) dz$$

$$= \int_0^1 z^2 \, dz$$

$$= \frac{1}{3} z^3 \Big|_0^1$$

$$= \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3$$

$$= \boxed{\frac{1}{3}}$$

Properties of the triple integral

① $$\iiint_E [f(x,y,z)+g(x,y,z)]\,dV = \iiint_E f(x,y,z)\,dV + \iiint_E g(x,y,z)\,dV$$

② If c is a constant, then

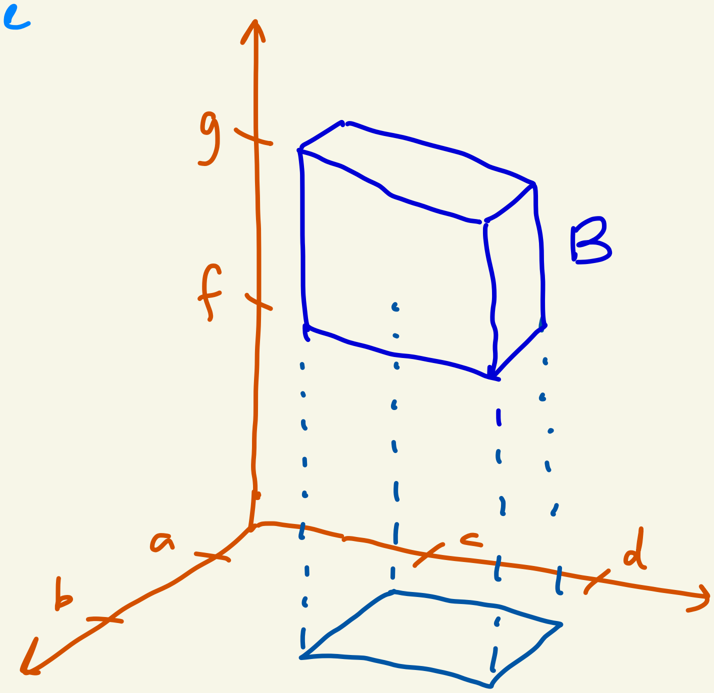
$$\iiint_E c\cdot f(x,y,z)\,dV = c\left[\iiint_E f(x,y,z)\,dV\right]$$

③ To calculate the volume of E, integrate $f(x,y,z)=1$ over E.

That is,

$$\text{volume}(E) = \iiint_E 1\,dV$$

<u>Ex:</u> Use the triple integral to find the volume of the box B defined by $a \le x \le b$, $c \le y \le d$, $f \le z \le g$.



The volume is

$$\iiint_B 1\, dV = \int_f^g \int_c^d \int_a^b 1\, dV$$

$$= \int_f^g \int_c^d \int_a^b 1\, dx\, dy\, dz$$

$$= \int_f^g \int_c^d \left[x \Big|_a^b \right] dy\, dz$$

$$= \int_f^g \int_c^d (b-a)\, dy\, dz$$

$$= (b-a) \left[\int_f^g \int_c^d 1\, dy\, dz \right]$$

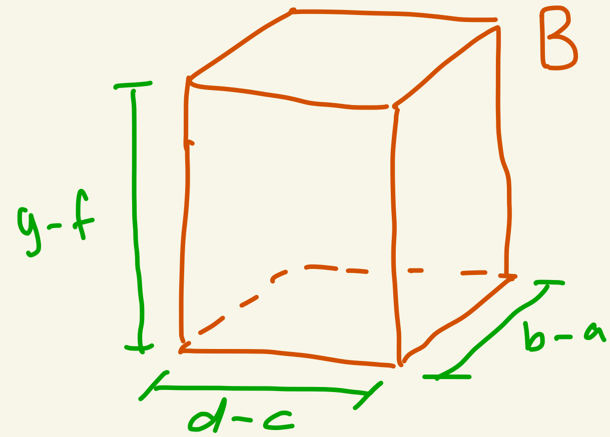
$$= (b-a) \int_f^g \left[y \Big|_{y=c}^{d} \right] dz$$

$$= (b-a) \int_f^g (d-c)\, dz$$

$$= (b-a)(d-c) \int_f^g 1\, dz$$

$$= (b-a)(d-c) \left[z \Big|_{z=f}^{g} \right]$$

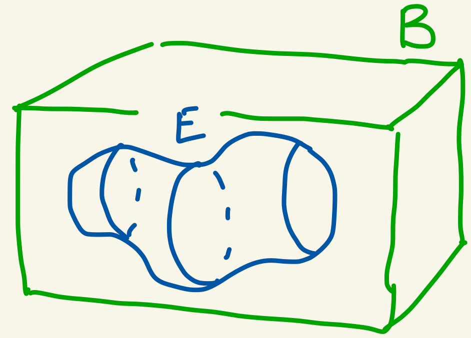
$$= \underbrace{(b-a)}_{\text{depth}} \underbrace{(d-c)}_{\text{width}} \underbrace{(g-f)}_{\text{height}}$$



Note: To generalize to integrals over any bounded region E, as with double integrals, put E inside a box and define $g(x,y,z)$ to equal $f(x,y,z)$ inside E and 0 outside E.

And define

$$\iiint_E f(x,y,z)\,dV = \iiint_B g(x,y,z)\,dV$$

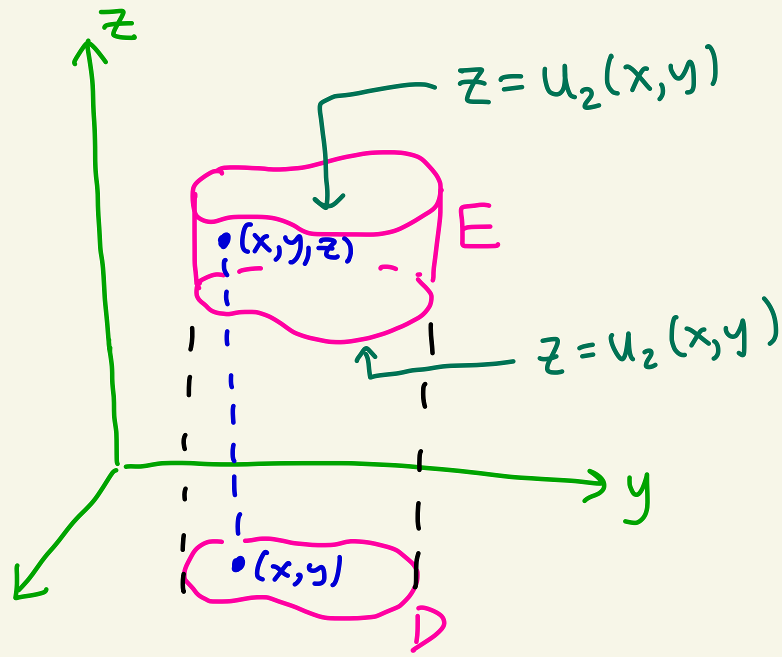


How do we calculate the triple integral over more general E?

Suppose E is described by all points (x,y,z) where

$u_1(x,y) \le z \le u_2(x,y)$

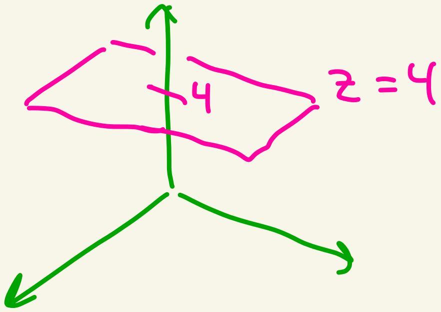
and where (x,y,z) all lie above D, the projection of E into the xy-plane.

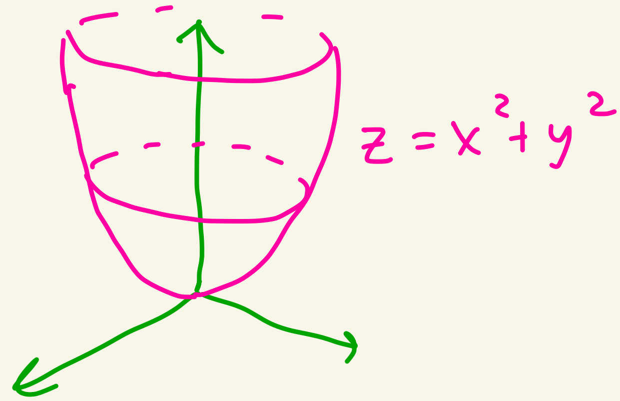


Then,

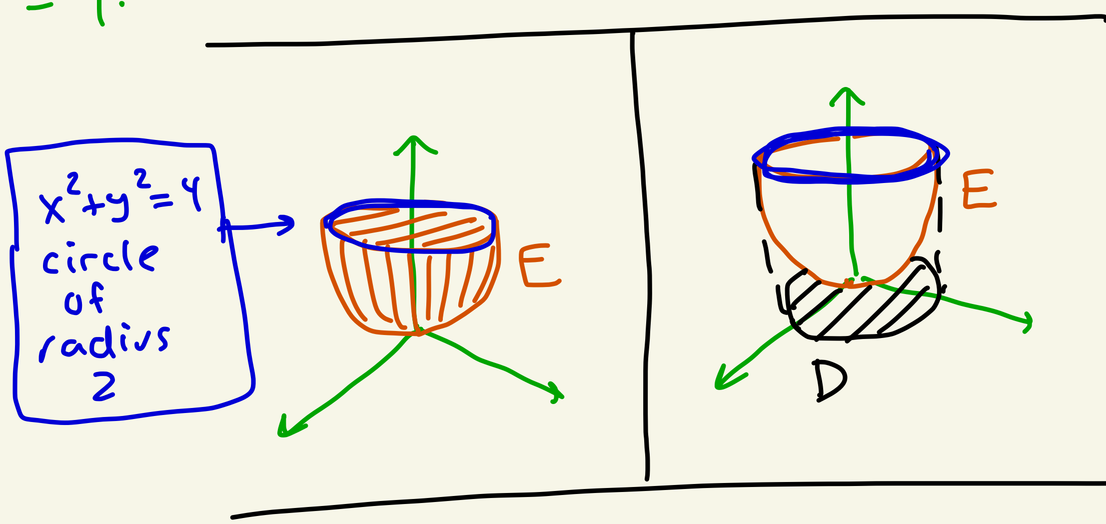
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

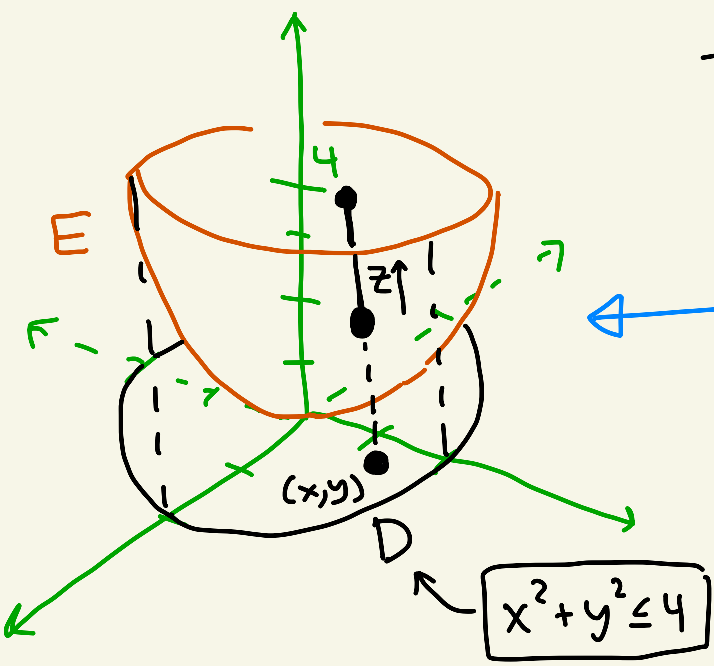
Ex: Find the volume of the solid E that lies between $z=4$ and $z=x^2+y^2$.





Intersection of $z=4$ and $z=x^2+y^2$ occurs when $x^2+y^2=4$.





note that (x,y) goes through all the points in D and for each of these we have $x^2+y^2 \le z \le 4$.

Thus, the volume is

$$\iiint_E 1 \, dV = \iint_D \left[\int_{x^2+y^2}^{4} 1 \, dz \right] dA$$

$$= \iint_D \left[z \Big|_{z=x^2+y^2}^{4} \right] dA$$

$$= \iint_D \left[4 - (x^2+y^2) \right] dA$$

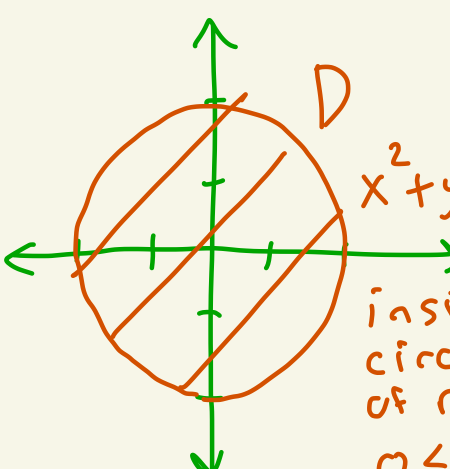
convert to polar coordinates

$$= \int_0^{2\pi} \int_0^2 \left[4 - r^2 \right] r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[4 \cdot \frac{r^2}{2} - \frac{r^4}{4} \right] \Bigg|_{r=0}^{2} d\theta$$

$$= \int_0^{2\pi} \left[\left(4 \cdot \frac{2^2}{2} - \frac{2^4}{4} \right) - (0) \right] d\theta$$



$x^2+y^2 \le 4$

inside circle of radius 2

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} (8-4)\, d\theta$$

$$= \int_0^{2\pi} 4\, d\theta$$

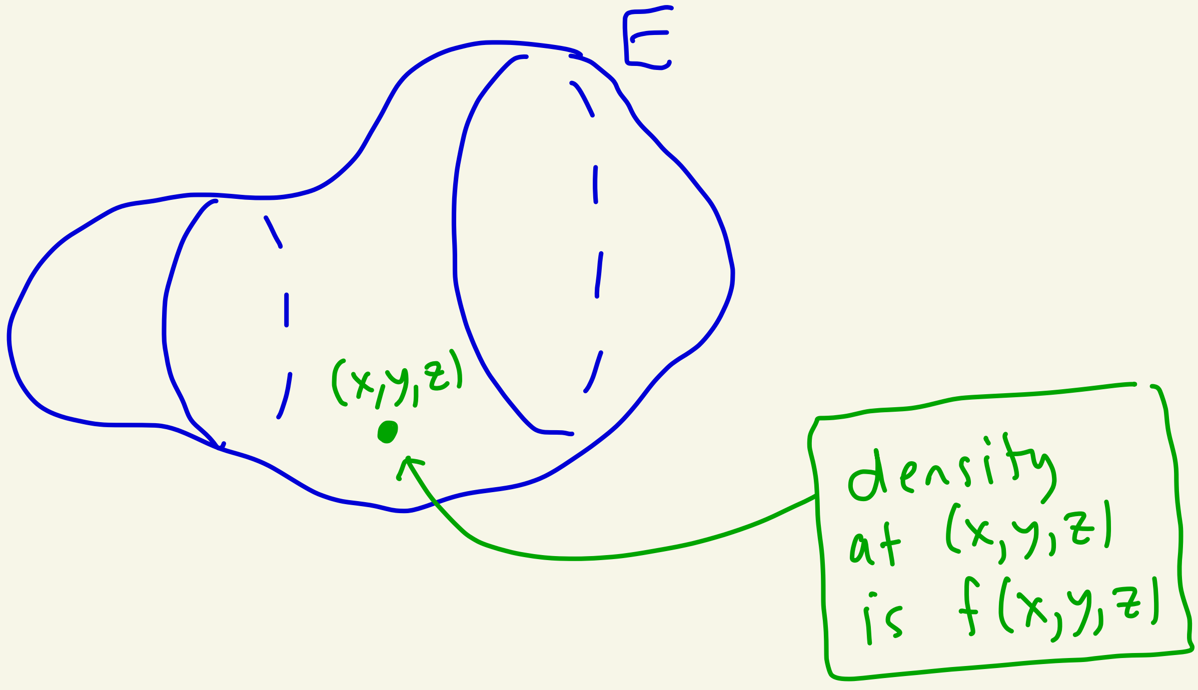
$$= 4\theta \Big|_{\theta=0}^{2\pi}$$

$$= 4(2\pi - 0)$$

$$= \boxed{8\pi}$$

Note: In physics, if the density of E at any point is given by $f(x,y,z)$ then the mass of E is

$$\text{mass of } E = \iiint_E f(x,y,z)\, dV$$



You can also project E into either the yz-plane or xz-plane if that is easier.

The formulas are similar to the one when projecting into the xy-plane

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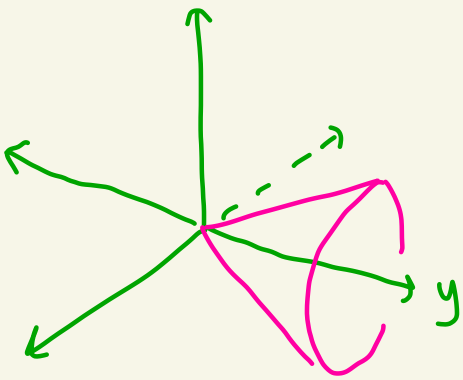
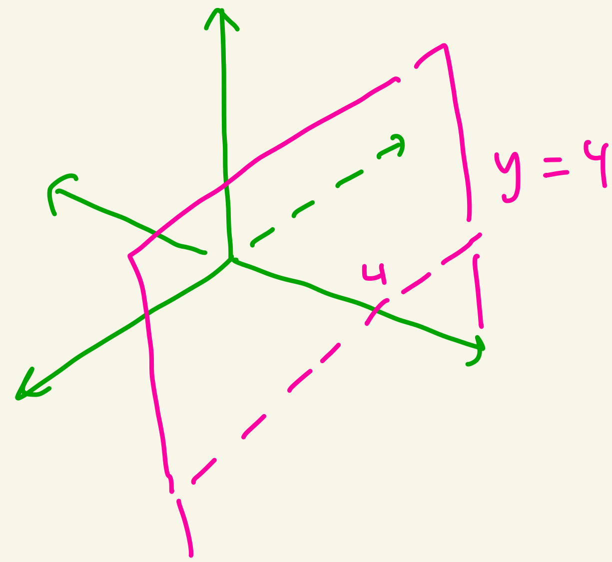
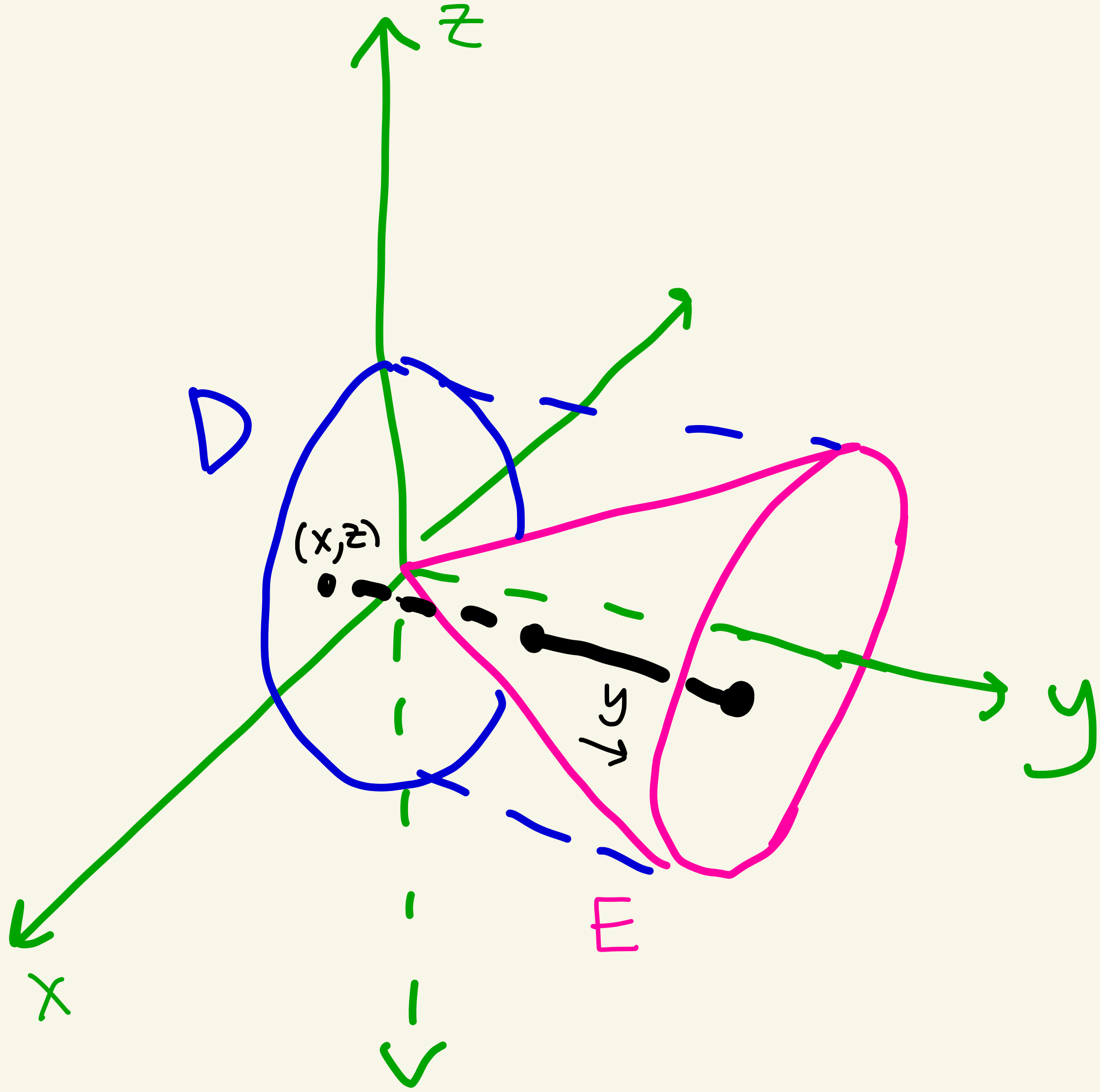
Ex: Let E be the solid that lies between the cone $y^2 = x^2 + z^2$ and $y = 2$.
Suppose at each point the density of E is given by $\rho(x,y,z) = 2y$.
Find the mass of E.



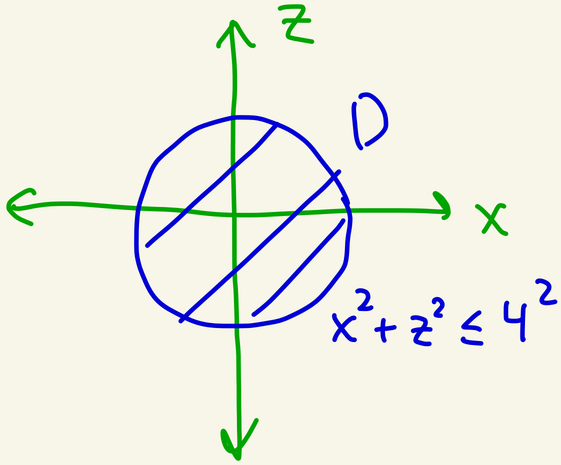


this top intersection of $y=4$ and $y^2 = x^2 + z^2$ occurs when $4^2 = x^2 + y^2$ at a circle of radius 4



As (x,z) go through D we have

$$\sqrt{x^2+z^2} \le y \le 4$$



We have

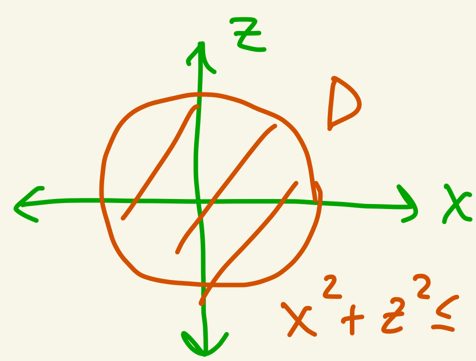
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{\sqrt{x^2+z^2}}^{4} 2y\,dy\right] dA$$

$$= \iint_D \left[y^2 \Big|_{y=\sqrt{x^2+z^2}}^{y=4} \right] dA$$

$$= \iint_D \left[4^2 - \left(\sqrt{x^2+z^2}\right)^2 \right] dA$$

$$= \iint_D \left[16 - (x^2+z^2)\right] dA$$

$$= \int_0^{2\pi} \int_0^4 \underbrace{\left[16 - r^2\right] r}_{16r - r^3}\, dr\, d\theta$$



$x^2+z^2 \le 4^2$

$x = r\cos(\theta)$
$z = r\sin(\theta)$
$r^2 = x^2+z^2$
$0 \le r \le 4$
$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \left(8r^2 - \frac{1}{4}r^4 \Big|_0^4\right) d\theta$$

$$= \int_0^{2\pi} \left[\left(8\cdot 4^2 - \frac{1}{4}\cdot 4^4\right) - (0-0)\right] d\theta$$

$$= \int_0^{2\pi} 64 \, d\theta$$

$$= 64\theta \Big|_0^{2\pi}$$

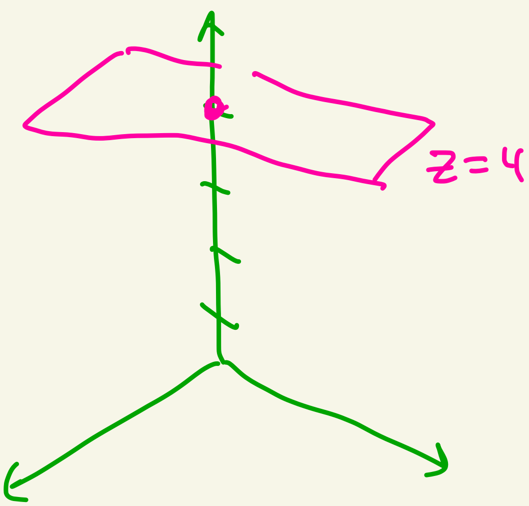
$$= 64(2\pi - 0)$$

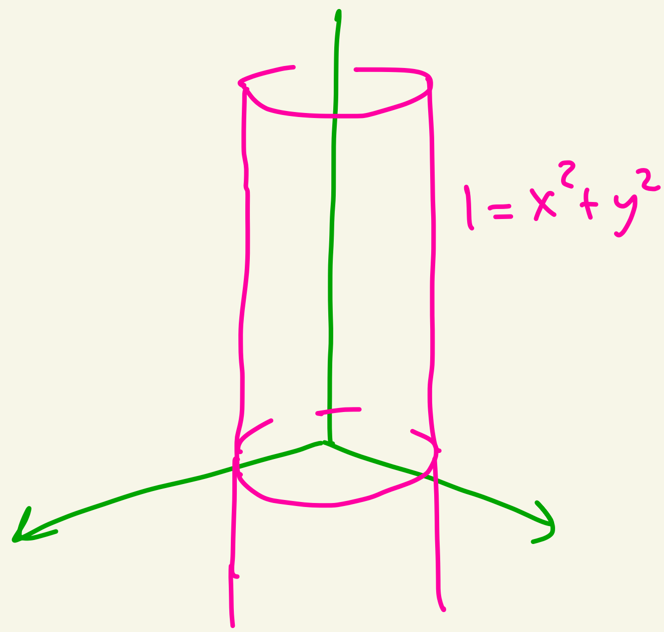
$$= \boxed{128\pi}$$

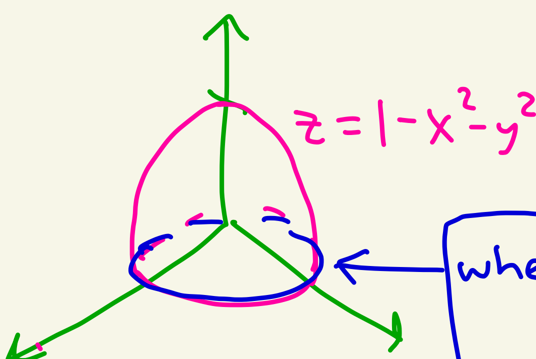
Ex: Let E be the solid that lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$.

Let the density at each point be $f(x,y,z) = 2\sqrt{x^2+y^2}$

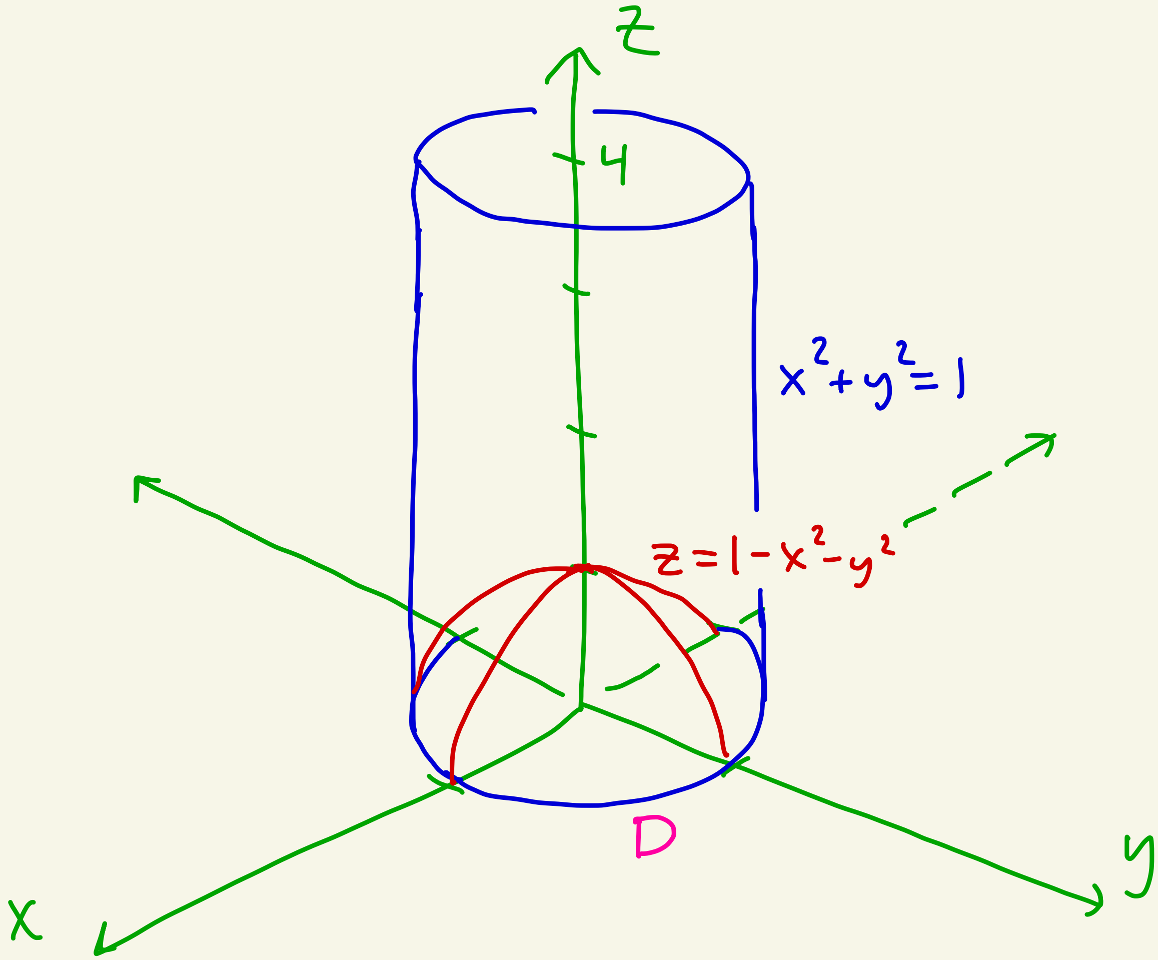
Find the mass of the solid E.

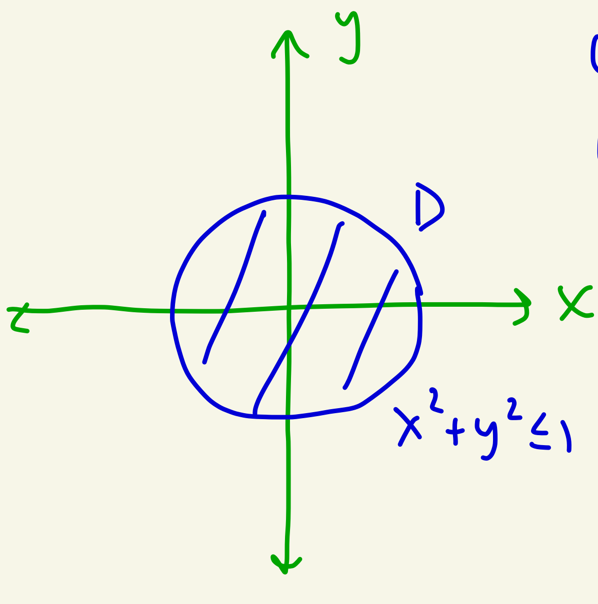






$\boxed{\begin{array}{l}\text{when } z=0 \text{ get } 0=1-x^2-y^2 \\ \qquad\qquad\qquad\qquad 1=x^2+y^2\end{array}}$





$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

The mass is

$$\iiint_E 2\sqrt{x^2+y^2}\, dV = \iint_D \left[\int_{1-x^2-y^2}^{4} 2\sqrt{x^2+y^2}\, dz \right] dA$$

$$= \iint_D \left[2\sqrt{x^2+y^2} \cdot z \Big|_{z=1-x^2-y^2}^{4} \right] dA$$

$$= \iint_D 2\sqrt{x^2+y^2} \left[4 - (1-x^2-y^2) \right] dA$$

$$= \int_0^{2\pi} \int_0^1 2\sqrt{r^2} \left[4 - (1-r^2) \right] r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \underbrace{\left(2r(3+r^2) r \right)}_{6r^2+2r^4} dr\, d\theta$$

$$= \int_0^{2\pi} \left(6\frac{r^3}{3} + 2\frac{r^5}{5} \right) \Big|_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi} \left(2 + \frac{2}{5}\right) - (0) \; d\theta$$

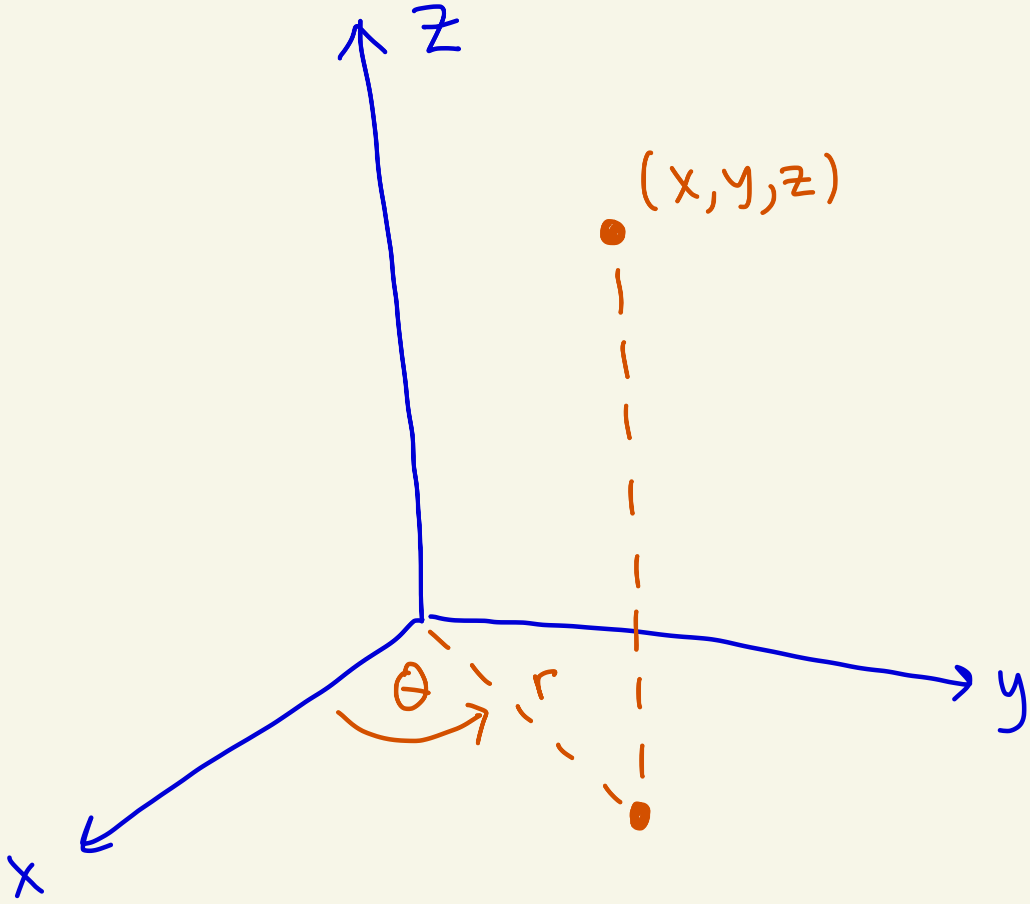
$$= \int_0^{2\pi} \frac{12}{5} \, d\theta$$

$$= \frac{12}{5} \theta \Big|_0^{2\pi}$$

$$= \frac{12}{5} (2\pi - 0)$$

$$= \boxed{\frac{24\pi}{5}}$$

<u>Note:</u> The above two examples illustrate "cylindrical coordinates" which are given as follows:



cylindrical coordinates

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$
$$x^2 + y^2 = r^2$$